

On $\mathcal{N} = 2$ supersymmetric gauge theories on $S^2 \times S^2$

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Abstract

We construct a supergeometry based on $S^2 \times S^2$ on which four dimensional $\mathcal{N} = 2$ gauge theories can be placed supersymmetrically while preserving all supersymmetries. By embedding the supergeometry in four dimensional $\mathcal{N} = 2$ supergravity we are able to construct an arbitrary $\mathcal{N} = 2$ gauge theory on $S^2 \times S^2$. We show that $\mathcal{N} = 2$ gauge theories are invariant under the exceptional superalgebra $D(2, 1, \alpha)$, where α is the ratio of the radii of the two S^2 's. We solve the supersymmetry fixed points equations for a choice of supercharge in $D(2, 1, \alpha)$. The solution of these BPS equations, which we find, would serve as the exact saddle point configurations of a localization computation of the partition function of $\mathcal{N} = 2$ gauge theories on $S^2 \times S^2$.

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1 Introduction

The computation of the partition function of four dimensional $\mathcal{N} = 2$ gauge theories on S^4 by Pestun [1] has led to new insights into the non-perturbative dynamics of gauge theories. A natural avenue of investigation is to consider more general curved backgrounds over which a four dimensional $\mathcal{N} = 2$ gauge theory can be placed, and to compute the corresponding exact partition function. Just as the S^4 partition function of $\mathcal{N} = 2$ superconformal field theories computes the exact Kähler potential on the conformal manifold [2] [3], it is interesting to understand the intrinsic physical meaning of the partition function of such theories on other backgrounds.

In this paper we identify a supergeometry based on $S^2 \times S^2$ over which an arbitrary four dimensional $\mathcal{N} = 2$ gauge theory can be placed while preserving all supercharges. The theory is constructed by embedding our $S^2 \times S^2$ background in four dimensional $\mathcal{N} = 2$ supergravity (supersymmetric backgrounds in $\mathcal{N} = 2$ supergravity have been considered in [4] [5]). We show that the theory is invariant under the exceptional superalgebra $D(2, 1, \alpha)$, where α is the ratio of the radii of the two S^2 's. We solve the supersymmetric fixed point equations for a choice of supercharge in $D(2, 1, \alpha)$. We find that the non-singular field configurations are labeled by quantized magnetic flux over each of the two S^2 's and have no continuous moduli. We also show that point-like instanton and anti-instanton configurations are supersymmetric at poles on $S^2 \times S^2$.

The plan of the rest of the paper is as follows. In section 2 we construct a supergeometry on $S^2 \times S^2$ by specifying Killing spinor equations and identify the supergeometry with a coset superspace. In section 3 we embed the Killing spinor equations in section 2 in four dimensional $\mathcal{N} = 2$ supergravity. This requires finding the off-shell field configurations for the supergravity multiplet. We also identify $D(2, 1, \alpha)$ as the supersymmetry algebra of four dimensional $\mathcal{N} = 2$ gauge theories on $S^2 \times S^2$. In section 4 we write down the supersymmetry transformations of the matter multiplets on this background. In section 5 we solve the supersymmetry fixed point equations for a choice of supercharge in $D(2, 1, \alpha)$, which would serve as the exact saddle points in the localization computation of $\mathcal{N} = 2$ gauge theories on $S^2 \times S^2$. Various computational details are relegated to the Appendices. Recently a paper studying the topologically twisted theory on $S^2 \times S^2$ has appeared in [6].

2 Supergeometry on $S^2 \times S^2$

Supersymmetry transformations in $\mathcal{N} = 2$ theories are parametrized by Killing spinors ϵ^i and ϵ_i of opposite chirality

$$\gamma_* \epsilon^i = \epsilon^i \quad \gamma_* \epsilon_i = -\epsilon_i \quad (2.1)$$

transforming as doublets of the $SU(2)_R$ R-symmetry (see Appendix A for conventions and notations).

On the background geometry $S^2 \times S^2$ of radii \tilde{r} and r we define the following consistent set of Killing spinor equations

$$\begin{aligned}\nabla_m \epsilon^i &= \frac{1}{2\tilde{r}} \Gamma \gamma_m \epsilon^{ij} \epsilon_j, & \nabla_m \epsilon_i &= \frac{1}{2\tilde{r}} \Gamma \gamma_m \epsilon_{ij} \epsilon^j & m &= 0, 1 \\ \nabla_p \epsilon^i &= \frac{i}{2r} \Gamma \gamma_p \epsilon^{ij} \epsilon_j, & \nabla_p \epsilon_i &= \frac{i}{2r} \Gamma \gamma_p \epsilon_{ij} \epsilon^j & p &= 2, 3\end{aligned}\quad (2.2)$$

where $\Gamma \equiv \gamma_{\hat{0}} \gamma_{\hat{1}}$. These equations can be diagonalized by combining the Killing spinors into $SU(2)_R$ doublets

$$\chi^i = \epsilon^i + i\epsilon^{ij} \epsilon_j, \quad (2.3)$$

such that

$$\epsilon^i = \chi_L^i \quad \epsilon_i = i\epsilon_{ij} \chi_R^j. \quad (2.4)$$

These obey

$$\begin{aligned}\nabla_m \chi^i &= -\frac{i}{2\tilde{r}} \gamma_m \Gamma \chi^i & m &= 0, 1 \\ \nabla_p \chi^i &= \frac{1}{2r} \gamma_p \Gamma \chi^i & p &= 2, 3.\end{aligned}\quad (2.5)$$

By choosing the following basis of γ -matrices

$$\gamma_{\hat{0}} = -\tau_2 \otimes 1 \quad (2.6)$$

$$\gamma_{\hat{1}} = \tau_1 \otimes 1 \quad (2.7)$$

$$\gamma_{\hat{2}} = \tau_3 \otimes \tau_1 \quad (2.8)$$

$$\gamma_{\hat{3}} = \tau_3 \otimes \tau_2 \quad (2.9)$$

$$\gamma_* = -\gamma_{\hat{0}} \gamma_{\hat{1}} \gamma_{\hat{2}} \gamma_{\hat{3}} = \tau_3 \otimes \tau_3, \quad (2.10)$$

the Killing spinor equations (2.2) decouple between the two S^2 's

$$\nabla_{\hat{0}} \chi^i = \frac{i}{2\tilde{r}} (\tau_1 \otimes 1) \chi^i \quad (2.11)$$

$$\nabla_{\hat{1}} \chi^i = \frac{i}{2\tilde{r}} (\tau_2 \otimes 1) \chi^i \quad (2.12)$$

$$\nabla_{\hat{2}} \chi^i = \frac{i}{2r} (1 \otimes \tau_1) \chi^i \quad (2.13)$$

$$\nabla_{\hat{3}} \chi^i = \frac{i}{2r} (1 \otimes \tau_2) \chi^i. \quad (2.14)$$

In the vielbien frame

$$e^{\hat{0}} = \tilde{r} d\tilde{\theta} \quad e^{\hat{1}} = \tilde{r} \sin \tilde{\theta} d\tilde{\phi} \quad e^{\hat{2}} = r d\theta \quad e^{\hat{3}} = r \sin \theta d\phi \quad (2.15)$$

we have that²

$$\chi^i = e^{\frac{i}{2}\tau_1 \tilde{\theta}} e^{\frac{i}{2}\tau_3 \tilde{\phi}} \otimes e^{\frac{i}{2}\tau_1 \theta} e^{\frac{i}{2}\tau_3 \phi} \chi_{(0)}^i \quad (2.16)$$

²Killing spinors on S^2 can be found in [7, 8].

where $\chi_{(0)}^i$ is a constant $SU(2)_R$ doublet of Dirac spinors. In Appendix B we express the spinors in the stereographic coordinate system and show that the spinors are non-singular everywhere on $S^2 \times S^2$.

The Killing spinors we have constructed are acted on by the $SU(2)_1 \times SU(2)_2$ isometries of $S^2 \times S^2$ through the Lie-Lorentz derivative. For a Killing vector field ξ this derivative acts by

$$\mathcal{L}_\xi = \nabla_\xi + \frac{1}{4} \nabla_\mu \xi_\nu \gamma^{\mu\nu}. \quad (2.17)$$

This analysis implies that the Killing spinors transform in the $(2, 2, 2)$ representation of $SU(2)_1 \times SU(2)_2 \times SU(2)_R$. The associated supergeometry is the coset superspace

$$\frac{D(2, 1, \alpha)}{U(1) \times U(1)}, \quad (2.18)$$

where $\alpha = \frac{\tilde{r}}{r}$.

3 $\mathcal{N} = 2$ Supergravity Background Fields for $S^2 \times S^2$

$\mathcal{N} = 2$ gauge theories on $S^2 \times S^2$ are based on a vectormultiplet and a hypermultiplet. Supersymmetry requires non-minimal couplings of the vectormultiplet and hypermultiplet to the background geometry. These can be found by the Noether procedure starting from the supersymmetry transformations and action of the theory in flat space.

A less laborious and more conceptual way of proceeding is to embed the supergeometry we have just constructed as a supersymmetric background of off-shell $\mathcal{N} = 2$ supergravity, in the spirit advocated in [9]. This approach relies on the already known supersymmetry transformations and couplings of a vectormultiplet and hypermultiplet to an off-shell supergravity multiplet. For our construction we consider the coupling of a vectormultiplet and hypermultiplet to the $\mathcal{N} = 2$ Weyl multiplet [10] (we refer to [11] for more details).

Off-shell $\mathcal{N} = 2$ superconformal transformations are realized on the Weyl multiplet, whose independent fields are

$$\begin{aligned} &\text{bosonic: } e_\mu^a, b_\mu, V_{\mu i}^j, A_\mu^R, T_{ab}, \bar{T}_{ab}, D \\ &\text{fermionic: } \psi_\mu^i, \psi_{\mu i}, \chi^i, \chi_i. \end{aligned} \quad (3.1)$$

The fields $e_\mu^a, b_\mu, V_{\mu i}^j, A_\mu^R, \psi_\mu^i, \psi_{\mu i}$ are the gauge fields for translations, dilatations, $SU(2)_R$, $U(1)_R$ and Poincaré supersymmetry generators in the $\mathcal{N} = 2$ superconformal algebra. The Weyl multiplet also includes the bosonic auxiliary fields T_{ab}, \bar{T}_{ab} and D , and the fermionic auxiliary fields χ^i and χ_i . In Euclidean signature T_{ab} is a self-dual and \bar{T}_{ab} is an anti-self-dual rank-two tensor.

Supersymmetric (bosonic) backgrounds are background values of the Weyl multiplet obeying

$$(\delta_\epsilon + \delta_\eta) \psi_\mu^i = 0 \quad (\delta_\epsilon + \delta_\eta) \chi^i = 0, \quad (3.2)$$

where (ϵ^i, ϵ_i) and (η^i, η_i) parametrize the Poincaré and conformal supersymmetry transformations.³ The explicit form of these transformations are [12] [13] [14] (we use [15])

$$\begin{aligned}\delta\psi_\mu^i &= \left(\partial_\mu + \frac{1}{4}\gamma^{ab}w_{\mu ab} - i\frac{1}{2}A_\mu^R + \frac{1}{2}b_\mu\right)\epsilon^i + V_\mu^i{}_j\epsilon^j - \frac{1}{16}\gamma^{ab}T_{ab}\epsilon^{ij}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i, \\ \delta\chi^i &= \frac{1}{2}D\epsilon^i + \frac{1}{6}\gamma^{ab}\left[-\frac{1}{4}\not{T}_{ab}\epsilon^{ij}\epsilon_j - \widehat{R}_{ab}(U_j^i)\epsilon^j + i\widehat{R}_{ab}(T)\epsilon^i + \frac{1}{2}T_{ab}^-\epsilon^{ij}\eta_j\right],\end{aligned}\quad (3.3)$$

where \mathcal{D} is the superconformal covariant derivative and $\widehat{R}_{ab}(T)$ and $\widehat{R}_{ab}(U_j^i)$ are covariant curvatures for $U(1)_R$ and $SU(2)_R$.

Our goal is to embed the Killing spinor equations on $S^2 \times S^2$ in (2.2) as a supersymmetric background for the Weyl multiplet. By analyzing (3.3) we find indeed that the following background fields give rise to our supersymmetric $S^2 \times S^2$ supergeometry

$$e_m^a = e_m^a|_{S^2 \times S^2}, \quad T_{\hat{0}\hat{1}} = T_{\hat{2}\hat{3}} = \left(\frac{i}{r} + \frac{1}{\tilde{r}}\right), \quad \bar{T}_{\hat{0}\hat{1}} = -\bar{T}_{\hat{2}\hat{3}} = \left(\frac{i}{r} + \frac{1}{\tilde{r}}\right), \quad D = \frac{1}{6}\left(\frac{1}{r^2} + \frac{1}{\tilde{r}^2}\right). \quad (3.4)$$

From (3.3) we find that the conformal supersymmetry parameters that give rise to $S^2 \times S^2$ are

$$\eta^i = \frac{1}{4}\left(\frac{i}{r} - \frac{1}{\tilde{r}}\right)\Gamma\epsilon^{ij}\epsilon_j, \quad \eta_i = \frac{1}{4}\left(\frac{i}{r} - \frac{1}{\tilde{r}}\right)\Gamma\epsilon_{ij}\epsilon^j. \quad (3.5)$$

The supergravity approach also provides us with a systematic way of identifying the superisometry algebra of a supersymmetric background of supergravity. The structure constants of the $\mathcal{N} = 2$ superconformal transformations generated by the closure of the supergravity transformations determine the rigid supersymmetry algebra of our $S^2 \times S^2$ background. This is obtained by evaluating the structure constants on the $S^2 \times S^2$ background fields (3.4)(3.5). The supergravity commutators yield⁴

$$[\delta_1, \delta_2] = \xi^m P_m + \lambda_a R^a + \lambda_D D + \lambda_R R + \frac{1}{2}\lambda^{ab} L_{ab}, \quad (3.6)$$

where $\delta \equiv \delta_\epsilon + \delta_\eta$. The parameters $(\xi^m, \lambda_a, \lambda_D, \lambda^{ab})$ are completely determined by the off-shell supergravity transformations of the Weyl multiplet.

Using the $S^2 \times S^2$ Killing spinor equations (2.2), it follows that the vector field produced by two superconformal transformations

$$\xi_m = \frac{1}{2}\bar{\epsilon}_2^i \gamma_m \epsilon_{1i} + \frac{1}{2}\bar{\epsilon}_{2i} \gamma_m \epsilon_1^i \quad (3.7)$$

³The supersymmetry transformations are the same as those in Lorentzian signature, but now (ϵ_i, ϵ^i) and (η^i, η_i) are not related by conjugation, and are independent spinors. Also, we allow all fields (except the metric) to be complex. In Euclidean signature, however, T_{ab} is selfdual while \bar{T}_{ab} is anti-selfdual, and are independent fields (see Appendix A).

⁴We omit special conformal transformations, as they act trivially on vectormultiplet and hypermultiplet fields. Actually, only the dilatation gauge field b_μ is acted on by special conformal transformations.

is a Killing vector on $S^2 \times S^2$. Using the Killing spinors (2.16) we compute ξ_m in Appendix C.

Evaluating the parameters in (3.6) on the $S^2 \times S^2$ background we find that dilatations D and the $U(1)_R$ R-symmetry are broken, while the $SU(2)_R$ R-symmetry is unbroken

$$\begin{aligned}\lambda_D &= -\frac{1}{2} (\bar{\epsilon}_1^i \eta_{2i} + \bar{\epsilon}_{1i} \eta_2^i - \bar{\epsilon}_2^i \eta_{1i} - \bar{\epsilon}_{2i} \eta_1^i) = 0, \\ \lambda_R &= \frac{i}{2} (\bar{\epsilon}_1^i \eta_{2i} - \bar{\epsilon}_{1i} \eta_2^i - \bar{\epsilon}_2^i \eta_{1i} + \bar{\epsilon}_{2i} \eta_1^i) = 0, \\ \lambda_j^i &= -\bar{\epsilon}_1^i \eta_{2j} + \bar{\epsilon}_{1j} \eta_2^i + \bar{\epsilon}_2^i \eta_{1j} - \bar{\epsilon}_{2j} \eta_1^i = \frac{1}{2} \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \left(-\varepsilon_{jk} \bar{\epsilon}_1^{(i} \Gamma \epsilon_2^{k)} + \bar{\epsilon}_{1(j} \Gamma \epsilon_{2k)} \varepsilon^{ik} \right). \quad (3.8)\end{aligned}$$

Therefore, we have shown starting from supergravity that the rigid supersymmetry algebra on our $S^2 \times S^2$ is the complexified $D(2, 1, \alpha)$ supersymmetry algebra, with $\alpha = \frac{\tilde{r}}{r}$. The eight conserved supercharges, which transform in the $(2, 2, 2)$ representation of $SU(2)_1 \times SU(2)_2 \times SU(2)_R$ close into the $SU(2)_1 \times SU(2)_2$ isometries of $S^2 \times S^2$ and the $SU(2)_R$ R-symmetry.⁵

4 $\mathcal{N} = 2$ Gauge Theories on $S^2 \times S^2$

Embedding $S^2 \times S^2$ as a supersymmetric background in $\mathcal{N} = 2$ supergravity allows us to immediately write down the $D(2, 1, \alpha)$ supersymmetry transformations acting on the vector-multiplet and hypermultiplet fields. These can be obtained from the supergravity literature by plugging in the $S^2 \times S^2$ background fields (3.4)(3.5) on the known superconformal supergravity transformations.

An off-shell $\mathcal{N} = 2$ vectormultiplet consists of

$$\begin{aligned}\text{bosonic: } & X, A_\mu, Y_{ij} \\ \text{fermionic: } & \Omega_i\end{aligned} \quad (4.1)$$

a complex scalar X , a gauge field A_μ , a triplet of real auxiliary fields $Y_{ij} = Y_{ji}$ and gauginos Ω_i . All fields in the multiplet transform in the adjoint representation of a gauge group G . An on-shell $\mathcal{N} = 2$ hypermultiplet consists of

$$\begin{aligned}\text{bosonic: } & q_i \\ \text{fermionic: } & \psi\end{aligned} \quad (4.2)$$

a doublet of scalars q_i and hyperinos ψ . The hypermultiplet can be coupled to a vector-multiplet by embedding the gauge group G in the symplectic symmetry group acting on the hypermultiplets. Fields in the hypermultiplet transform in a representation R of G . We consider the vectormultiplet and a hypermultiplet coupled to the Weyl multiplet.

⁵On the fields, a local Lorentz transformation is also induced.

On a vectormultiplet and hypermultiplet the commutator of two supergravity transformations yields (3.6) *together* with a gauge transformation acting in the appropriate representation of the gauge group G . The induced field dependent gauge transformation parameter is

$$\Lambda = X\epsilon^{ij}\bar{\epsilon}_{2i}\epsilon_{1j} + \bar{X}\epsilon_{ij}\bar{\epsilon}_2^i\epsilon_1^j, \quad (4.3)$$

which when written in terms of the doublets (2.3) is

$$\Lambda = \frac{1}{2}(\bar{X} - X)\epsilon_{ij}\bar{\chi}_2^i\chi_1^j + \frac{1}{2}(X + \bar{X})\epsilon_{ij}\bar{\chi}_2^i\gamma_*\chi_1^j. \quad (4.4)$$

This gauge transformation plays an important role in the computation of the partition function of $\mathcal{N} = 2$ gauge theories on $S^2 \times S^2$.

The supersymmetry transformations and invariant action for the vectormultiplet and hypermultiplet can be obtained from Chapter 20 of [11]. For future reference, the $D(2, 1, \alpha)$ supersymmetry transformations acting on the gauginos in the vectormultiplet are

$$\delta\Omega_i = \frac{1}{4} \left[F_{ab} - \frac{1}{2}\bar{X}T_{ab} \right] \gamma^{ab}\epsilon_{ij}\epsilon^j + \gamma^\mu D_\mu X \epsilon_i - i[X, \bar{X}]\epsilon_{ij}\epsilon^j + Y_{ij}\epsilon^j + 2X\eta_i \quad (4.5)$$

$$\delta\Omega^i = \frac{1}{4} \left[F_{ab} - \frac{1}{2}X\bar{T}_{ab} \right] \gamma^{ab}\epsilon^{ij}\epsilon_j + \gamma^\mu D_\mu \bar{X} \epsilon^i + i[X, \bar{X}]\epsilon^{ij}\epsilon_j + Y^{ij}\epsilon_j + 2\bar{X}\eta^i. \quad (4.6)$$

(ϵ^i, ϵ_i) are the Killing spinors on $S^2 \times S^2$ we constructed, (η^i, η_i) are given in (3.5) and T_{ab} and \bar{T}_{ab} in (3.4).

The action of the vectormultiplets and action and on-shell supersymmetry transformations for the hypermultiplet are given in chapters 20.2.4 and 20.2.3 of [11] by substituting the background fields (3.4) on $S^2 \times S^2$.

5 Supersymmetric Fixed Points on $S^2 \times S^2$

In this final section we find the supersymmetric field configurations associated to a particular supersymmetry transformation in $D(2, 1, \alpha)$. These field configurations correspond to the exact saddle points of the partition function of $\mathcal{N} = 2$ gauge theories on $S^2 \times S^2$ when computed by supersymmetric localization with the corresponding supercharge.

The supersymmetry transformation that we consider is generated by the following choice of constant spinor $\chi_{(0)}^j$ in (2.16)

$$\chi_{(0)}^j = \delta_1^j \left[\frac{1 + \tau_3}{2} \otimes \frac{1 + \tau_3}{2} \right] \chi_A^j + \delta_2^j \left[\frac{1 - \tau_3}{2} \otimes \frac{1 - \tau_3}{2} \right] \chi_B^j, \quad (5.1)$$

which projects onto the sum $(\uparrow, \uparrow, \uparrow) \oplus (\downarrow, \downarrow, \downarrow)$ under $SU(2)_1 \times SU(2)_2 \times SU(2)_R$. The corresponding transformation obeys

$$\delta^2 = \tilde{J}_3 + J_3 + R, \quad (5.2)$$

where \tilde{J}_3 and J_3 are the Cartan generators of the $SU(2)$ isometry acting on $S^2_{\tilde{r}}$ and S^2_r respectively, while R is the $SU(2)_R$ R-symmetry Cartan generator. The equivariant parameters induced by the supersymmetry transformation for \tilde{J}_3 and J_3 are given by

$$\varepsilon_1 = \frac{i}{\tilde{r}} \quad \varepsilon_2 = \frac{1}{\tilde{r}} \quad (5.3)$$

while the equivariant parameter for $SU(2)_R$ is

$$\varepsilon_1 + \varepsilon_2. \quad (5.4)$$

On the vectormultiplet and hypermultiplet the induced gauge transformation is

$$\Lambda = (\bar{X} - X) \cos \theta + (X + \bar{X}) \cos \tilde{\theta}. \quad (5.5)$$

The Killing spinor corresponding to the choice (5.1) is non-chiral everywhere on $S^2 \times S^2$ except at the four fixed points of δ^2 , labeled by a North and/or South pole for each of the S^2 's. At these four fixed points the non-vanishing Killing spinor has a definite chirality:

$$\text{NN} : L \quad \text{NS} : R \quad \text{SN} : R \quad \text{SS} : L. \quad (5.6)$$

Since chiral (anti-chiral) supersymmetry transformations correspond to instanton (anti-instanton) field configurations, this implies that on $S^2 \times S^2$ there are supersymmetric point-like instanton configurations at NN and SS poles and supersymmetric point-like anti-instanton configurations at the NS and SN poles.

We now analyse the supersymmetric fixed point equations for our choice of supersymmetry transformation (5.1). We have already identified singular instanton and anti-instanton field configurations at the poles of $S^2 \times S^2$, so we now turn to the analysis of the smooth supersymmetric field configurations. This requires solving the equations

$$\delta\Omega^i = \delta\Omega_i = 0 \quad (5.7)$$

for our choice of transformation. We write the supersymmetry equations in Appendix D. The most general smooth solution is labeled by a pair of vector of integers (\tilde{B}, B) that represent quantized flux over each of the two S^2 's. These fluxes take values in the Cartan subalgebra of the gauge group G . The supersymmetry equations also fix the scalar field in the vectormultiplet in terms of the fluxes. Explicitly⁶

$$F_{0\tilde{1}} = \frac{2}{\tilde{r}} \text{Re}(X) = \frac{\tilde{B}}{2\tilde{r}^2} \quad F_{23} = \frac{2}{r} \text{Im}(X) = \frac{B}{2r^2} \quad (5.8)$$

⁶This is similar to the Coulomb branch localization saddle points of two dimensional $\mathcal{N} = (2, 2)$ gauge theories on S^2 [7, 8].

In solving the supersymmetry equations we have used the standard reality properties on the fields, in particular $X^\dagger = \bar{X}$. As in the case of S^4 [1] and squashed S^4 [4], there are no non-trivial solutions to the supersymmetry equations for the fields in the hypermultiplet.

Our analysis suggests the following answer for the partition of $\mathcal{N} = 2$ gauge theories on $S^2 \times S^2$. It is given by the sum over all quantized fluxes B and \tilde{B} of the product the instanton partition function at the NN and SS poles with the anti-instanton partition function at the NS and SN poles. These Omega-background instanton partition functions [16] are to be evaluated for the values of the equivariant parameters induced by the $S^2 \times S^2$ geometry. The geometrical ones are governed by (5.3). The equivariant gauge transformation is obtained by evaluating (5.5) on the supersymmetric field configurations (5.8). When evaluated on the poles, we get

$$\Lambda_{NN} = -\Lambda_{SS} = 2\bar{X} \quad \Lambda_{NS} = -\Lambda_{SN} = 2X \quad (5.9)$$

and therefore the equivariant gauge parameters are quantised. It would be interesting to confirm this intuition by a detailed supersymmetric localization computation

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A Notations and Conventions

Curved indices are μ, ν, \dots while tangent space indices are a, b, \dots . We also denote tangent space indices with a hat. e.g., $\hat{0}$. We split $\mu = (m, p)$, where $m = 0, 1$ parametrizes S_r^2 and $p = 2, 3$ parametrizes S_r^2 .

We take the Lorentzian chirality matrix to be

$$\gamma_* = i\gamma_{\hat{0}}\gamma_{\hat{1}}\gamma_{\hat{2}}\gamma_{\hat{3}} = -i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}}. \quad (\text{A.1})$$

We continue to Euclidean signature by changing coordinates

$$x^0 = -ix_E^0, \quad (\text{A.2})$$

so that

$$\gamma_{\hat{0}E} = -i\gamma_{\hat{0}} \quad \gamma_E^{\hat{0}} = i\gamma^{\hat{0}} \quad (\text{A.3})$$

This implies that Euclidean chirality matrix is

$$\gamma_* = -\gamma_{\hat{0}E}\gamma_{\hat{1}E}\gamma_{\hat{2}E}\gamma_{\hat{3}E} = -\gamma^{\hat{0}E}\gamma^{\hat{1}E}\gamma^{\hat{2}E}\gamma^{\hat{3}E}. \quad (\text{A.4})$$

We drop the index E to avoid cluttering. The chirality of the various fermions is

SUSY							gravity multiplet				
ϵ^i	ϵ_i	η^i	η_i	Q^i	Q_i	S^i	S_i	ψ_μ^i	$\psi_{\mu i}$	χ^i	χ_i
L	R	R	L	R	L	L	R	L	R	L	R

A L and R chiral fermion obeys

$$P_L\psi = \psi = \gamma_*\psi \quad P_R\psi = \psi = -\gamma_*\psi \quad (\text{A.5})$$

where

$$P_L = \frac{1}{2}(1 + \gamma_*) \quad P_R = \frac{1}{2}(1 - \gamma_*). \quad (\text{A.6})$$

Epsilon tensor: Defined to obey $\epsilon^{12} = \epsilon_{12} = 1$. It satisfies

$$\epsilon^{ik}\epsilon_{kj} = -\delta_j^i. \quad (\text{A.7})$$

B Killing Spinors in stereographic coordinates

The metric on $S^2 \times S^2$ in stereographic coordinates reads

$$ds^2 = 4\tilde{r}^2 \frac{dw d\bar{w}}{(1 + |w|^2)^2} + 4r^2 \frac{dz d\bar{z}}{(1 + |z|^2)^2}.$$

The stereographic coordinates (w, z) cover a patch including the north pole of each S^2 and are given in terms of the spherical coordinates by

$$\begin{aligned} w &= \tan \frac{\tilde{\theta}}{2} e^{i\tilde{\phi}} = \frac{1}{u} \\ z &= \tan \frac{\theta}{2} e^{i\phi} = \frac{1}{v}. \end{aligned}$$

Coordinates (u, v) cover a patch including the south pole of each S^2 .

Vielbeins regular at the (N,N) poles ($w = 0, z = 0$) are

$$\hat{e}^0 = \frac{2\tilde{r}}{1 + \bar{w}w} dw_R, \quad \hat{e}^1 = \frac{2\tilde{r}}{1 + \bar{w}w} dw_I, \quad \hat{e}^2 = \frac{2r}{1 + \bar{z}z} dz_R, \quad \hat{e}^3 = \frac{2r}{1 + \bar{z}z} dz_I. \quad (\text{B.1})$$

In terms of the original ones they are written as an $SO(2) \times SO(2)$ rotation, given by

$$\begin{aligned} \hat{e}^0 &= \cos \tilde{\phi} e^0 - \sin \tilde{\phi} e^1 & \hat{e}^2 &= \cos \phi e^2 - \sin \phi e^3 \\ \hat{e}^1 &= \sin \tilde{\phi} e^0 + \cos \tilde{\phi} e^1 & \hat{e}^3 &= \sin \phi e^2 + \cos \phi e^3. \end{aligned}$$

Under such a change of frame, the Killing spinors transform under an $SO(2) \times SO(2)$ rotation as spinors. Hence

$$\hat{\chi}^i = \exp \left(-\frac{\tilde{\phi}}{2} \gamma_{\hat{0}\hat{1}} \right) \exp \left(-\frac{\phi}{2} \gamma_{\hat{2}\hat{3}} \right) \chi^i$$

where

$$\chi^i = e^{\frac{i}{2}\tau_1\tilde{\theta}} e^{\frac{i}{2}\tau_3\tilde{\phi}} \otimes e^{\frac{i}{2}\tau_1\theta} e^{\frac{i}{2}\tau_3\phi} \chi_0^i$$

and

$$\begin{aligned} \gamma_{\hat{0}\hat{1}} &= i\tau_3 \otimes I \\ \gamma_{\hat{2}\hat{3}} &= iI \otimes \tau_3. \end{aligned}$$

We need to calculate

$$e^{-\frac{i}{2}\tau_3\phi} e^{\frac{i}{2}\tau_1\theta} e^{\frac{i}{2}\tau_3\phi} = \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} e^{-i\phi} \\ i \sin \frac{\theta}{2} e^{i\phi} & \cos \frac{\theta}{2} \end{pmatrix} = \cos \frac{\theta}{2} \begin{pmatrix} 1 & i\bar{z} \\ iz & 1 \end{pmatrix} \quad (\text{B.2})$$

$$(\text{B.3})$$

and likewise for the other S^2 . Note that the prefactor is non-vanishing around the corresponding poles, and the matrix elements combine into regular functions of stereographic coordinates. This implies that the spinors around all poles are regular.

For the $(\uparrow, \uparrow, \uparrow)$ spinor we have (tensored with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$) for the $SU(2)_R$ R-symmetry)

$$\hat{\chi}^1 = \frac{1}{\sqrt{(1 + \bar{w}w)(1 + \bar{z}z)}} \begin{pmatrix} 1 \\ iz \\ iw \\ -wz \end{pmatrix}, \quad (\text{B.4})$$

while for the $(\downarrow, \downarrow, \downarrow)$ spinor we have (tensored with $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$) for the $SU(2)_R$ R-symmetry)

$$\hat{\chi}^2 = \frac{1}{\sqrt{(1 + \bar{w}w)(1 + \bar{z}z)}} \begin{pmatrix} -\bar{w}\bar{z} \\ i\bar{w} \\ i\bar{z} \\ 1 \end{pmatrix}. \quad (\text{B.5})$$

We have used that

$$\cos \frac{\theta}{2} = \frac{1}{\sqrt{1 + z\bar{z}}} \quad \sin \frac{\theta}{2} = \frac{1}{\sqrt{1 + v\bar{v}}}. \quad (\text{B.6})$$

C Killing Vectors from Killing Spinors

The Killing vector obtained by two SUSY transformations is (using that $\bar{\epsilon}\gamma^\alpha\lambda = -\bar{\lambda}\gamma^\alpha\epsilon$)

$$\frac{1}{2}\bar{\epsilon}_2^i\gamma^a\epsilon_{1i} + \frac{1}{2}\bar{\epsilon}_{2i}\gamma^a\epsilon_1^i = -\frac{i}{4}[\bar{\chi}_{2+}^i\gamma^a\epsilon_{ij}\chi_1^j - \bar{\chi}_{2-}^i\gamma^a\epsilon_{ij}\chi_{1+}^j] \quad (\text{C.1})$$

$$= \frac{i}{2}\bar{\chi}_2^i\gamma_*\gamma^a\epsilon_{ij}\chi_1^j. \quad (\text{C.2})$$

They are (computation of $\bar{\chi}_2^i\gamma_*\gamma^\mu\epsilon_{ij}\chi_1^j\partial_\mu = e_a^\mu\bar{\chi}_2^i\gamma_*\gamma^a\epsilon_{ij}\chi_1^j\partial_\mu$)

$$0 : \left[i \cos \tilde{\phi} \bar{\chi}_0^i (\tau_1 \otimes \tau_3) \epsilon_{ij} \chi_0^j + i \sin \tilde{\phi} \bar{\chi}_0^i (\tau_2 \otimes \tau_3) \epsilon_{ij} \chi_0^j \right] \frac{\partial_{\tilde{\theta}}}{\tilde{r}} \quad (\text{C.3})$$

$$1 : \left[i \cot \tilde{\theta} \cos \tilde{\phi} \bar{\chi}_0^i (\tau_2 \otimes \tau_3) \epsilon_{ij} \chi_0^j - i \cot \tilde{\theta} \sin \tilde{\phi} \bar{\chi}_0^i (\tau_1 \otimes \tau_3) \epsilon_{ij} \chi_0^j + i \bar{\chi}_0^i (\tau_3 \otimes \tau_3) \epsilon_{ij} \chi_0^j \right] \frac{\partial_{\tilde{\phi}}}{\tilde{r}} \quad (\text{C.4})$$

$$2 : \left[i \cos \phi \bar{\chi}_0^i (1 \otimes \tau_2) \epsilon_{ij} \chi_0^j - i \sin \phi \bar{\chi}_0^i (1 \otimes \tau_1) \epsilon_{ij} \chi_0^j \right] \frac{\partial_\theta}{r} \quad (\text{C.5})$$

$$3 : \left[-i \cot \theta \cos \phi \bar{\chi}_0^i (1 \otimes \tau_1) \epsilon_{ij} \chi_0^j - i \cot \theta \sin \phi \bar{\chi}_0^i (1 \otimes \tau_2) \epsilon_{ij} \chi_0^j + \bar{\chi}_0^i (1 \otimes 1) \epsilon_{ij} \chi_0^j \right] \frac{\partial_\phi}{r} \quad (\text{C.6})$$

These are precisely the six Killing vectors of $S_r^2 \times S_{\tilde{r}}^2$.

D BPS equations

The BPS equations associated to the spinor $\chi_{(0)}^i = (\uparrow, \uparrow, \uparrow) + (\downarrow, \downarrow, \downarrow)$ are explicitly

$$\begin{aligned} (1 + \gamma_*) \left[Y_{11} \chi_+^1 + A \left[\frac{1}{2} X \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \Gamma + i \not{D} X + \frac{1}{4} \gamma^{ab} \mathcal{F}_{ab} - i [X, \bar{X}] + Y_{12} \right] \chi_+^2 \right] &= 0 \\ (1 + \gamma_*) \left[\left[\frac{1}{2} X \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \Gamma + i \not{D} X + \frac{1}{4} \gamma^{ab} \mathcal{F}_{ab} - i [X, \bar{X}] - Y_{21} \right] \chi_+^1 - A Y_{22} \chi_+^2 \right] &= 0 \end{aligned}$$

and

$$\begin{aligned} (1 - \gamma_*) \left[A Y^{11} \chi_+^2 - \left[\frac{1}{2} \bar{X} \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \Gamma + i \not{D} \bar{X} + \frac{1}{4} \gamma^{ab} \mathcal{F}_{ab} + i [X, \bar{X}] + Y^{12} \right] \chi_+^1 \right] &= 0 \\ (1 - \gamma_*) \left[A \left[\frac{1}{2} \bar{X} \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \Gamma + i \not{D} \bar{X} + \frac{1}{4} \gamma^{ab} \mathcal{F}_{ab} + i [X, \bar{X}] - Y^{21} \right] \chi_+^2 + Y^{22} \chi_+^1 \right] &= 0 \end{aligned}$$

For the spinor $\chi_{(0)}^i = (\uparrow, \uparrow, \uparrow) + (\downarrow, \downarrow, \downarrow)$ given in appendix (B), they read as

$$\begin{aligned} -\bar{w} \bar{z} ((-i\mathcal{F}_{\hat{0}\hat{1}} - i\mathcal{F}_{\hat{2}\hat{3}}) - i \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) X + 2i [X, \bar{X}] + 2Y_{12}) - (-i\mathcal{F}_{\hat{0}\hat{2}} - \mathcal{F}_{\hat{0}\hat{3}} - \mathcal{F}_{\hat{1}\hat{2}} + i\mathcal{F}_{\hat{1}\hat{3}}) &+ \\ 2\bar{z} (D_{\hat{1}} + iD_{\hat{0}}) X + 2\bar{w} (D_{\hat{2}} - iD_{\hat{3}}) X - iY_{11} &= 0 \\ -\bar{w} \bar{z} ((i\mathcal{F}_{\hat{0}\hat{2}} - \mathcal{F}_{\hat{0}\hat{3}} - \mathcal{F}_{\hat{1}\hat{2}} - i\mathcal{F}_{\hat{1}\hat{3}})) - \left[-i\mathcal{F}_{\hat{0}\hat{1}} - i\mathcal{F}_{\hat{2}\hat{3}} - i \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) X - 2i [X, \bar{X}] - 2Y_{12} \right] &+ \\ 2\bar{w} (D_{\hat{1}} - iD_{\hat{0}}) X - 2\bar{z} (D_{\hat{2}} + iD_{\hat{3}}) X + iwzY_{11} &= 0 \\ (-i\mathcal{F}_{\hat{0}\hat{1}} - i\mathcal{F}_{\hat{2}\hat{3}}) - i \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) X + 2i [X, \bar{X}] + 2Y_{12} + zw(-i\mathcal{F}_{\hat{0}\hat{2}} - \mathcal{F}_{\hat{0}\hat{3}} - \mathcal{F}_{\hat{1}\hat{2}} + i\mathcal{F}_{\hat{1}\hat{3}}) &+ \\ 2w(D_{\hat{1}} + iD_{\hat{0}}) X + 2z(D_{\hat{2}} - iD_{\hat{3}}) X - i\bar{w}\bar{z}Y_{22} &= 0 \\ (i\mathcal{F}_{\hat{0}\hat{2}} - \mathcal{F}_{\hat{0}\hat{3}} - \mathcal{F}_{\hat{1}\hat{2}} - i\mathcal{F}_{\hat{1}\hat{3}}) + zw \left[-i\mathcal{F}_{\hat{0}\hat{1}} - i\mathcal{F}_{\hat{2}\hat{3}} - i \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) X - 2i [X, \bar{X}] - 2Y_{12} \right] &+ \\ 2z(D_{\hat{1}} - iD_{\hat{0}}) X - 2w(D_{\hat{2}} + iD_{\hat{3}}) X + iY_{22} &= 0 \end{aligned}$$

and

$$\begin{aligned} w(-i\mathcal{F}_{\hat{0}\hat{2}} + \mathcal{F}_{\hat{0}\hat{3}} - \mathcal{F}_{\hat{1}\hat{2}} - i\mathcal{F}_{\hat{1}\hat{3}}) + z \left[i\mathcal{F}_{\hat{0}\hat{1}} - i\mathcal{F}_{\hat{2}\hat{3}} + i \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \bar{X} + 2i [X, \bar{X}] - 2Y_{12} \right] &- \\ 2wz(D_{\hat{1}} + iD_{\hat{0}}) \bar{X} + 2(D_{\hat{2}} + iD_{\hat{3}}) \bar{X} - i\bar{w}Y_{11} &= 0 \\ w \left[-i\mathcal{F}_{\hat{0}\hat{1}} + i\mathcal{F}_{\hat{2}\hat{3}} - i \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \bar{X} + 2i [X, \bar{X}] - 2Y_{12} \right] + z(-i\mathcal{F}_{\hat{0}\hat{2}} - \mathcal{F}_{\hat{0}\hat{3}} + \mathcal{F}_{\hat{1}\hat{2}} - i\mathcal{F}_{\hat{1}\hat{3}}) &+ \\ 2(D_{\hat{1}} - iD_{\hat{0}}) \bar{X} + 2wz(D_{\hat{2}} - iD_{\hat{3}}) \bar{X} - i\bar{z}Y_{11} &= 0 \\ \bar{z}(-i\mathcal{F}_{\hat{0}\hat{2}} + \mathcal{F}_{\hat{0}\hat{3}} - \mathcal{F}_{\hat{1}\hat{2}} - i\mathcal{F}_{\hat{1}\hat{3}}) + \bar{w} \left[i\mathcal{F}_{\hat{0}\hat{1}} - i\mathcal{F}_{\hat{2}\hat{3}} + i \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \bar{X} + 2i [X, \bar{X}] - 2Y_{12} \right] &+ \\ 2(D_{\hat{1}} + iD_{\hat{0}}) \bar{X} - 2\bar{w}\bar{z}(D_{\hat{2}} + iD_{\hat{3}}) \bar{X} + i\bar{z}Y_{22} &= 0 \\ \bar{z} \left[-i\mathcal{F}_{\hat{0}\hat{1}} + i\mathcal{F}_{\hat{2}\hat{3}} - i \left(\frac{i}{r} - \frac{1}{\tilde{r}} \right) \bar{X} + 2i [X, \bar{X}] - 2Y_{12} \right] + \bar{w}(-i\mathcal{F}_{\hat{0}\hat{2}} - \mathcal{F}_{\hat{0}\hat{3}} + \mathcal{F}_{\hat{1}\hat{2}} - i\mathcal{F}_{\hat{1}\hat{3}}) &- \\ 2\bar{w}\bar{z}(D_{\hat{1}} - iD_{\hat{0}}) \bar{X} - 2(D_{\hat{2}} - iD_{\hat{3}}) \bar{X} + iwY_{22} &= 0 \end{aligned}$$

The non-vanishing field configurations are (5.8)

$$F_{\hat{0}\hat{1}} = \frac{2}{\tilde{r}} \text{Re}(X) = \frac{\tilde{B}}{2\tilde{r}^2} \quad F_{\hat{2}\hat{3}} = \frac{2}{r} \text{Im}(X) = \frac{B}{2r^2} . \quad (\text{D.1})$$

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